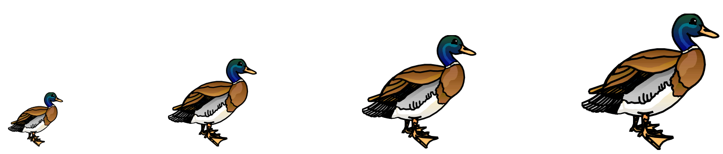
FACTORIALS

The odds of winning the original California lottery depended on the number of ways that **6** numbers could be chosen from a set of **51** numbers. Listing them would be ludicrous (since, as you'll discover in the chapter Combinations, there are over **18** million ways!). This chapter will develop the concept of the ***factorial*** function, leading us to a not-too-difficult formula for such a calculation.

❑ GETTING ALL OUR DUCKS IN A ROW



We start by posing the question:

How many ways are there to line up 4 ducks in a straight line?

We'll call the ducks a, b, c, and d. Let's start listing all the ways, organized in columns to keep track of what we've already listed.

abcd	*bacd*	*cadb*	*dabc*
abdc	*badc*	*cadb*	*dacb*
acbd	*bcad*	*cbad*	*dbac*
acdb	*bcda*	*cbda*	*dbca*
adbc	*bdac*	*cdab*	*dcab*
adcb	*bdca*	*cdba*	*dcba*

Make sure that we haven't omitted any, and also make sure we haven't included duplicates. It appears that we have **24** ways to line up the 4 ducks. That wasn't too bad, but now consider this: *How many ways are there to line up 10 ducks in a straight line*? My advice to you is don't bother — at 1 second per duck lineup, working 24 hours per day, it would take exactly 42 days to complete the list. What we need is a formula, based on some new insight, to describe what it really means to place 4 things in a row. We'll discover one in the following section.

Homework

1. How many ways are there to line up 1 ostrich in a straight line?
2. How many ways are there to line up 2 people in a straight line?
3. How many ways are there to line up 3 books in a straight line?
4. How many ways are there to line up 0 ducks in a straight line?

❑ *FACTORIAL*

Now we re-solve the duck question posed above:

How many ways are there to line up 4 ducks in a straight line?

using the chapter The Counting Principle.

How many ways are there to choose a duck to be first in line? There are **4** ways (since there are 4 ducks from which to choose).

Now, how many choices are there for the second position? There are **3**, because only 3 ducks remain.

For the third position, there are **2** choices because 2 ducks remain, and for the fourth position, there's only **1** choice because only 1 duck remains. Thus, there are $\mathbf{4 \times 3 \times 2 \times 1}$ ways of lining up the ducks, which comes to **24** — the same answer we obtained when we listed them by brute force in the chapter The Counting Principle.

"Because computations like $\mathbf{4 \times 3 \times 2 \times 1}$ occur frequently in mathematics, they are represented by a special notation. We say "four ***factorial***" and use an exclamation point to write "four factorial" as "**4!**". Thus,

$$\mathbf{4!} = 4 \times 3 \times 2 \times 1 = \mathbf{24}$$

Also, in the first section of this chapter, I claimed that it would take 42 days to line up 10 ducks. Let's see why: First, we know there are "10 factorial" ways to line up the 10 ducks. That's

$$\mathbf{10!} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{3{,}628{,}800}$$

ways to line up the 10 ducks. At 1 second per duck lineup, that's a total time of 3,628,800 seconds, which equals 60,480 minutes, which is 1,008 hours, which converts to exactly **42 days**. But we don't need 42 days to list them all to find out <u>how many ways</u> there are to line up the ducks — we just calculate **10!** and we have the number of ways.

Now we're ready for the official definition of factorial:

> If n is a natural number (1, 2, 3, . . .), then
>
> ***n factorial*** is defined to be
>
> $$n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)$$

<u>EXAMPLE 1:</u>

A. $3! = 3 \cdot 2 \cdot 1 = \mathbf{6}$

B. $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{5{,}040}$

C. $\dfrac{10!}{7! \times 3!} = \dfrac{3{,}628{,}800}{5{,}040 \times 6} = \dfrac{3{,}628{,}800}{30{,}240} = \mathbf{120}$

D. $\dfrac{200!}{198!}$ This is a tough one. Even with a calculator, multiplying 200 by 199 by 198 by 197 right on down the line would take a really long time. Or, we could use the "factorial" button on the calculator, but we'll likely get an ERROR message. So, what to do? Check this out:

$$\frac{200!}{198!} = \frac{200 \times 199 \times 198 \times 197 \times 196 \times \cdots \times 3 \times 2 \times 1}{198 \times 197 \times 196 \times \cdots \times 3 \times 2 \times 1}$$

See what cancels? All the factors from 198 down to 1.

$$= \frac{200 \times 199 \times \cancel{198 \times 197 \times 196 \times \ldots \times 3 \times 2 \times 1}}{\cancel{198 \times 197 \times 196 \times \ldots \times 3 \times 2 \times 1}}$$

All that remains is 200 × 199, which equals **39,800**. Cool?

E. $$\frac{75!}{72! \times 3!} = \frac{75 \times 74 \times 73 \times 72 \times 71 \times 70 \times \cdots \times 3 \times 2 \times 1}{[72 \times 71 \times 70 \times \cdots \times 3 \times 2 \times 1] \times [3 \times 2 \times 1]}$$

$$= \frac{75 \times 74 \times 73 \times \cancel{72 \times 71 \times 70 \times \cdots \times 3 \times 2 \times 1}}{[\cancel{72 \times 71 \times 70 \times \cdots \times 3 \times 2 \times 1}] \times [3 \times 2 \times 1]}$$

$$= \frac{75 \times 74 \times 73}{3 \times 2 \times 1}$$

$= 25 \times 37 \times 73$ [75/3 = 25 and 74/2 = 37]

$= \mathbf{67{,}525}$

F. $2! = 2 \times 1 = \mathbf{2}$

G. $1! = \mathbf{1}$

H. $0! = \mathbf{1}$

This result doesn't follow from the definition of factorial. We simply define **0!** to equal **1**. It's more of a philosophical question: How many ways are there to line up no ducks? Do you buy the notion that there's **1** way to do that?

But there's also a more concrete reason why we define 0! = 1. It makes future formulas (permutations and combinations) work perfectly.

Homework

5. Calculate: a. $(7 + 2)!$ b. $8! - 7!$ c. $2^{3!}$

6. Calculate: a. $\dfrac{97!}{95!}$ b. $\dfrac{100!}{95! \times 5!}$ c. $\dfrac{347!}{(0!)(347!)}$

7. You have to line up 20 kindergartners at the door in all possible ways. It takes one second for each possible lineup. How many years will it take you to accomplish this feat (assuming they cooperate)?

8. Does the factorial operation "respect addition"? That is, is the following statement true?

$$(a + b)! = a! + b!$$

Factorials with Variables: n! (n-1)! (n – 2)! (n + 3)!

Review Problems

Factorials

9. Calculate: $\dfrac{200!}{5!\times 195!}$

10. Calculate: $0! + 1! + 2! + 3! + 4! + 5!$

11. Analyze the following conjecture: $(ab)! = a!\,b!$

12. True/False:

    a. If a family of five ducks is trying to line up in all possible orders, but the papa duck must be in the first slot, then there are 24 ways the family of ducks can line up.

    b. There are 8! ways of lining up 8 children in a straight line.

    c. $\dfrac{1{,}000!}{500!} = 2$

    d. $0! = 0$

    e. $(m + n)! \neq m! + n!$ [where m and n are any whole numbers]

Solutions

1. 1 **2.** 2 **3.** 6

4. It's somewhat of a philosophical answer, but let's agree that there's 1 way.

5. a. 362,880 b. 35,280 c. 64

6. a. 9,312 b. 75,287,520 c. 1

7. About 77,000,000,000 years (77 billion years — vastly longer than the age of the universe).

8. Choose $a = 2$ and $b = 3$. Then

$$(a + b)! = (2 + 3)! = 5! = 120$$

$$a! + b! = 2! + 3! = 2 + 6 = 8$$ They're not even close.

Conclusion: $(a + b)! \neq a! + b!$

9. 2,535,650,040 **10**. 154

11. Let $a = 3$ and $b = 2$.
Then $(ab)! = (3 \cdot 2)! = 6! = 120$,
while $a!\,b! = 3!\,2! = 6 \times 2 = 12$.
The conjecture is false.

12. a. T b. T c. F d. F e. T

❑ *To ∞ and Beyond*

[Research] Find the sum of the series consisting of the reciprocals of the factorials:

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

which is the infinite sum $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ out to infinity.

"What the mind of man can conceive and believe, it can achieve."

Napoleon Hill